CHAPTER 16

# Celestial Coordinates

April Cheng

> *Mortal as I am, I know that I am born for a day. But when I follow at my pleasure the serried multitude of the stars in their circular course, my feet no longer touch the earth.*
>
> Ptolemy

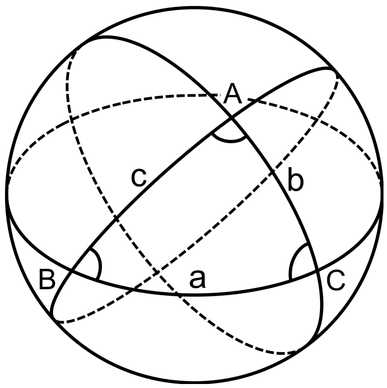
Perhaps the most long-standing tenet of astronomy, the concept of the celestial sphere—a spherical dome of sky encompassing the Earth upon which the sun and stars traverse—forms the foundation of observational astronomy. In this chapter, we leave temporarily the larger perspective that astronomy grants us and return to our perspective on Earth. Ptolemy and the rest of the pre-Copernican world believed in a geocentric universe. While we know that to be incorrect now, it is still useful to imagine a stationary Earth enclosed by a celestial sphere on which the heavens move. After all, we, and (most of) our telescopes, still touch the earth.

## 16.1 Spherical Trigonometry

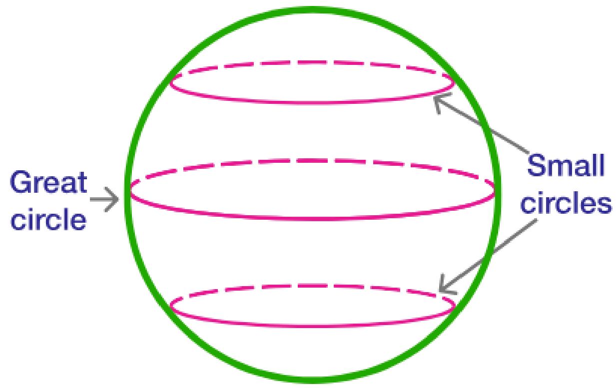
Intermediate

Many problems regarding the celestial sphere can be reduced to one or more **spherical triangle**s, which are triangles on the surface of a sphere formed by great-circle arcs (Figure 16.1a). A **great circle** is the largest possible circle that can be drawn on the surface of a sphere. The center of a great circle of a sphere must coincide with the center of the sphere (Figure 16.1b). Note that the side of a spherical triangle is not a length but rather an angle: the angle that the great circle arc subtends with respect to the center of the sphere. **The side and angle labeling scheme given in Figure 16.1a will be used for the rest of this section**.

There are two commonly used formulas in spherical trigonometry, the spherical law of cosines and the spherical law of sines, which are similar to their planar counterparts. They are used to prove many of the following concepts in this chapter.




(a) A spherical triangle, with conventional labels. (Image Credit: Wikipedia)




(b) The center of a great circle is the center of the sphere. All other circles are small circles. (Image Credit: Brilliant)

Figure 16.1: A spherical triangle (left), which is composed of arcs from great circles, as opposed to small circles (right).

**Theorem 16.1 (The Spherical Law of Cosines).** *For a spherical triangle with sides a, b, c and opposing vertices with angles A, B, C,*

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

*Alternatively,*

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

*Of course, this theorem would not change if you rotated all the labels, so it is also true that*

$$\cos b = \cos a \cos c + \sin a \sin c \cos B$$

*and so on.*

*Proof.* Consider a spherical triangle on a unit sphere. To analyze this triangle, let's choose an $xyz$ coordinate system with the origin $O$ at the center of the sphere, the $z$-axis going through vertex $A$, and the $x$-axis aligned with vertex $B$ (see Figure 16.2). We have three vectors $\vec{\alpha}$, $\vec{\beta}$, and $\vec{\gamma}$.

From the diagram, we can see that $\tilde{\alpha} = (0, 0, 1)$, since $\tilde{\alpha}$ lies on the $z$-axis on the surface of a unit sphere. We chose an x-axis such that $\tilde{\beta}$ would have no $y$-component; the $x$ and $z$ components are given by $\sin c$ and $\cos c$. Vector $\tilde{\gamma}$ is the most complex. It can be split into a $z$-component ($\cos b$) and a component in the $xy$ plane ($\sin b$). This can then be split into $x$ and $y$ components with $\angle A$, yielding $x$ and $y$ components $\sin b \cos A$ and $\sin b \sin A$, respectively.

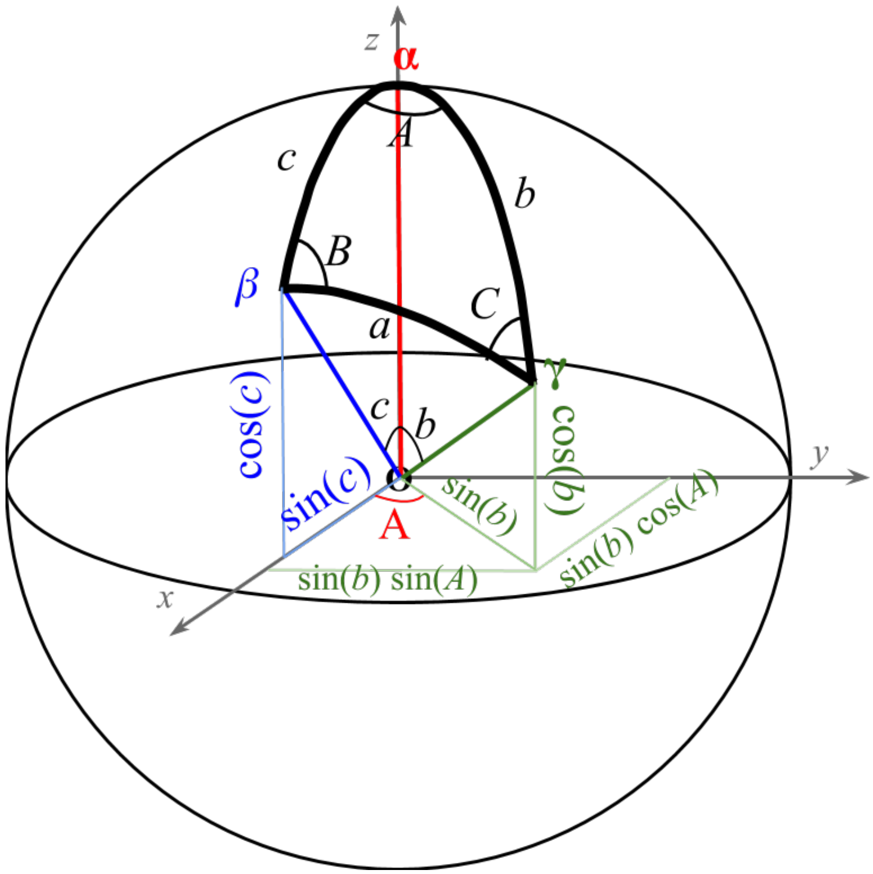



Figure 16.2: A spherical triangle and the associated $xyz$ coordinate system. The $xyz$ components of each of vectors $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ are labelled. (Adapted from Wikipedia)

Thus, we have:

$$
\begin{aligned}
\tilde{\alpha} &= (0, 0, 1) \\
\tilde{\beta} &= (\sin c, 0, \cos c) \\
\tilde{\gamma} &= (\sin b \cos A, \sin b \sin A, \cos b)
\end{aligned}
$$

We have two ways of computing the dot product $\tilde{\beta} \cdot \tilde{\gamma}$: 1) using $\tilde{\beta} \cdot \tilde{\gamma} = ||\tilde{\beta}||\,||\tilde{\gamma}|| \cos(\theta)$, where $\theta = a$ is the angle between vectors $\tilde{\beta}$ and $\tilde{\gamma}$, or 2) using $\tilde{\beta} \cdot \tilde{\gamma} = \beta_x \gamma_x + \beta_y \gamma_y + \beta_z \gamma_z$. The first method yields $\tilde{\beta} \cdot \tilde{\gamma} = \cos a$ (since they are both unit vectors) and the second method yields $\tilde{\beta} \cdot \tilde{\gamma} = \sin c \cdot \sin b \cos A + \cos c \cdot \cos b$. Equating the two yields our equation

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

which is the spherical law of cosines. $\square$

**Theorem 16.2 (The Spherical Law of Sines).** *For a spherical triangle with sides a, b, c and opposing vertices with angles A, B, C, the ratio of the sine of a side and the sine of the opposing angle is the same for all three sides:*

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

*Proof.* We start from the spherical law of cosines

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Substituting this into the identity $\sin^2 A + \cos^2 A = 1$ and doing some algebraic manipulation:

$$\begin{aligned}
\sin^2 A &= 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2 \\
&= \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c} \\
&= \frac{(1-\cos^2 b)(1-\cos^2 c) - (\cos^2(a) + \cos^2 b \cos^2 c - 2\cos a \cos b \cos c)}{\sin^2 b \sin^2 c} \\
&= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c - 2\cos a \cos b \cos c}{\sin^2 b \sin^2 c}
\end{aligned}$$

Taking the square root of both sides and dividing by $\sin a$,

$$\frac{\sin A}{\sin a} = \frac{(1 - \cos^2 a - \cos^2 b - \cos^2 c - 2\cos a \cos b \cos c)^{1/2}}{\sin a \sin b \sin c}$$

Notice that the right-hand side does not change if we rotate $a, b, c$. Thus, performing the same analysis for $B$ and $b$ and $C$ and $c$ would yield the same right-hand side. Therefore,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c},$$

which is the spherical law of sines. □

Advanced

There is one more formula I would like to include in this section; while not as commonly used, it has its applications in certain celestial coordinates astronomy problems with two consecutive sides and angles on a spherical triangle (such that the spherical laws of cosines and sines cannot be immediately used).

**Theorem 16.3 (The Cotangent Four-Part Formula).** *For any two sides and two angles that form four consecutive parts of a spherical triangle,*

$$\begin{aligned}
\cos(\text{inner side})\cos(\text{inner angle}) = {} & \cot(\text{outer side})\sin(\text{inner side}) - \\
& \cot(\text{outer angle})\sin(\text{inner angle})
\end{aligned}$$

*For example, for sides and angles A, b, C, and a, the cotangent four-part formula becomes*

$$\cos b \cos C = \cot a \sin b - \cot A \sin C$$

*There are 6 unique rotations of the cotangent four-part formula.*

*Proof.* We begin with the spherical law of cosines

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

For four consecutive sides, we want to get rid of the $c$ term and instead have our equation in terms of $C$. (Alternatively, we could do the same with $b$ and $B$.) We can substitute out $\cos c$ and $\sin c$ using the spherical laws of sines and cosines:

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \qquad\qquad \sin c = \sin C \frac{\sin b}{\sin B}$$

Substituting and doing some algebraic manipulation,

$$\begin{aligned}
\cos a &= \cos a \cos^2 b + \cos b \sin a \sin b \cos C + \sin b \sin C \sin a \cot A \\
\cos a \sin^2 b &= \cos b \sin a \sin b \cos C + \sin b \sin C \sin a \cot A \\
\cot a \sin b &= \cos b \cos C + \sin C \cot A \\
\cos b \cos C &= \cot a \sin b - \cot A \sin C.
\end{aligned}$$

We could do this analysis with any of the three spherical law of cosines, with two choices of side substitutions for each, yielding the six cotangent four-part formulae. □

Let's put these formulas into use with a problem:

**Example 16.1.** For the following spherical triangle (Figure 16.3), find expression(s) that would describe the quantity $A$ in terms of known quantities $\phi$, $H$, $\delta$, and $a$. $A$ may range from $0°$ to $360°$.

*Solution.* The spherical law of sines relates pairs of opposite sides and angles. Here, we do have two pairs we can apply the spherical law of sines to: $H$ and $a$, and $A$ and $\delta$.

$$\begin{aligned}
\frac{\sin(360° - A)}{\sin(90° - \delta)} &= \frac{\sin H}{\sin(90° - a)} \\
\sin A &= -\sin H \frac{\cos\delta}{\cos a}
\end{aligned}$$

Simply using arcsin on this equation would not be sufficient to find the value of $A$, since $A$ is not necessarily in the range of arcsin, $[-90°, 90°]$. For example, $\arcsin(\sqrt{2}/2) = 45°$, even though $\sin(225°)$ also equals $\sqrt{2}/2$. In other words, we don't know if the angle is in the left half of the unit circle or the right half. This is where we can use the spherical law of cosines to find $\cos A$!

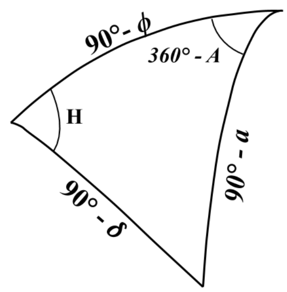



Figure 16.3: A spherical triangle. (Own Work)

The spherical law of cosines relates three sides and an angle:

$$\begin{aligned}\cos(90^\circ - \delta) &= \cos(90^\circ - a)\cos(90^\circ - \phi) + \\ &\quad \sin(90^\circ - a)\sin(90^\circ - \phi)\cos(360^\circ - A) \\ \cos A &= \frac{\sin\delta - \sin a \sin\phi}{\cos a \cos\phi}\end{aligned}$$

Finding both $\cos A$ and $\sin A$ allows us know which quadrant the angle is in, and thus the exact value of $A$ without any ambiguity.

The spherical triangle we used in this example and the equations we derived from it are actually important results in the topic of celestial coordinates! You will learn what the quantities on the spherical triangle represent and the meaning of these equations in the coming sections.

□

## 16.2 The Celestial Sphere

Beginner

As briefly mentioned in the introduction of this chapter, the **celestial sphere** is an imaginary sphere of an arbitrarily large (infinite) radius that surrounds the Earth. Everything—stars, planets, asteroids, galaxies, satellites—resides and moves on this celestial sphere. This imaginary celestial sphere is useful because knowing an object's position on the celestial sphere tells us where it is in the sky with respect to observers on Earth, so that we know where to point our eyes or our telescopes!

Before we begin, I would like to note that understanding the celestial sphere is greatly aided by visuals. I have tried to include a variety of different diagrams in this textbook, but please do not hesitate to search on Google images (or any equivalent) for more. In particular, HyperPhysics has great diagrams for the different celestial coordinate systems (horizontal, equatorial, ecliptic). Visualizing the celestial sphere in different ways helps a lot in understanding and eventually being able to visualize it in your own mind.

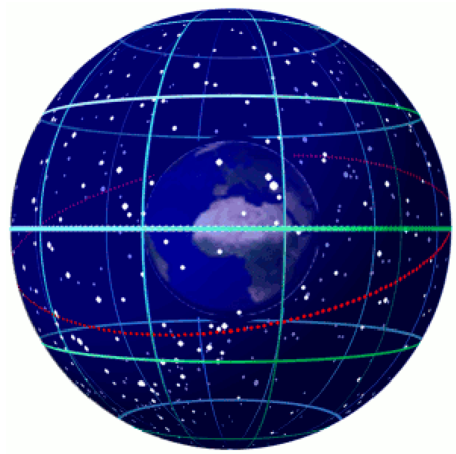


Figure 16.4: How one might imagine the celestial sphere. The cyan grid lines are of the equatorial coordinate system; the red great circle is the ecliptic (Sections 16.2.3, 16.2.3). (Image Credit: Wikipedia)

The celestial sky is populated with stars, and throughout history people have grouped stars into constellations: patterns of stars representing people, animals, and various objects. In modern astronomy, however, they hold no physical significance and are used instead as organizational bins. The International Astronomical Union (IAU) has split up the celestial sphere into 88 official constellations, many of which are the ancient Greek constellations documented by Ptolemy. Each IAU constellation has a strictly defined region that it occupies, and any deep sky objects, planets, or asteroids that fall within that region are said to be in that constellation. For example, the open cluster Messier 45 (the Pleiades) and all its stars, are in Taurus, even though it is arguably closer to Perseus or Aries if you were to judge by the constellation patterns alone (see Figure 16.5). This designation is also useful for naming stars and other objects (T Tauri, Cygnus X-1, Cassiopeia A, etc.).

Of course, constellations are insufficient to pinpoint the location of an object on the celestial sphere. We need a spherical coordinate system to map the sky!

### 16.2.1 The Horizontal Coordinate System

Beginner

You should already be familiar with at least one spherical coordinate coordinate system: Earth's! To map the Earth, we have assigned every point on Earth a spherical coordinate with latitude and longitude. We only need two things to define a latitude-longitude coordinate system: 1) a plane (or great circle) to choose as 0° latitude and 2) a reference point for 0° longitude. In Earth's case, we defined the equator to be our 0° latitude circle and the Greenwich meridian to be 0° latitude (see Figure 16.6(a)).

The most intuitive coordinate system for the celestial sphere is the **horizontal coordinate**

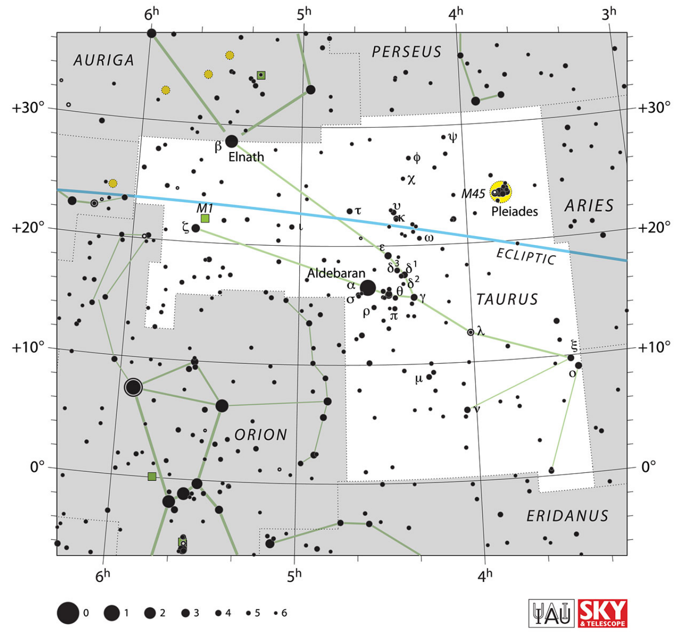
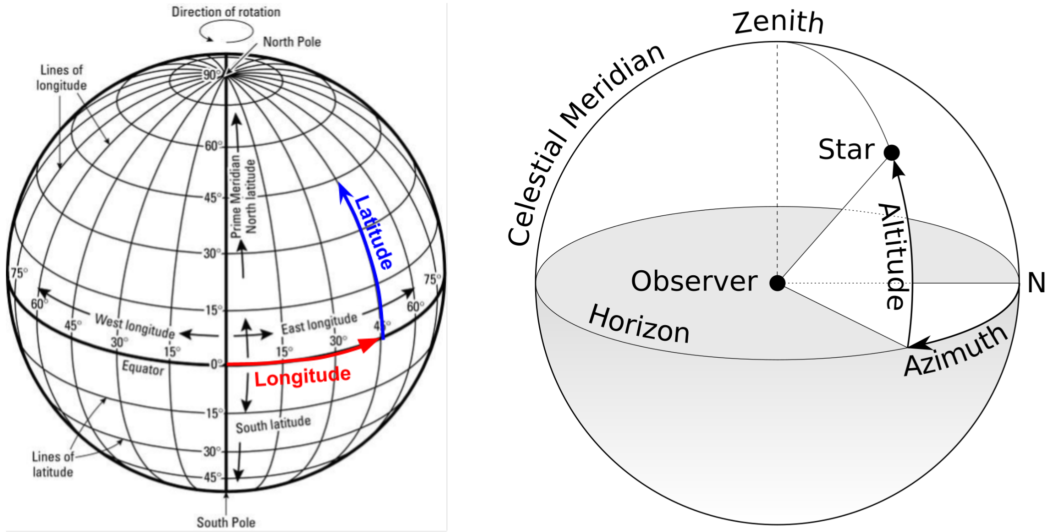



Figure 16.5: The constellation Taurus and its borders, as defined by the International Astronomical Union. The Pleiades are shown by the yellow circle in the top right. (Image Credit: IAU)

**system** (see Figure 16.6(b)). We define the coordinate system relative to the observer, such that the horizon is 0° "latitude" and North is 0° "longitude". Of course, it is not called latitude and longitude, but instead altitude and azimuth. Thus, the horizontal coordinate system is sometimes referred to as the **altitude-azimuth** (or alt/az) coordinate system. Formally, **altitude**, sometimes referred to as **elevation angle**, is the angle between the object and the observer's horizon. An object on the horizon (the setting sun, for example) would have an altitude of 0°, and any objects below the horizon would have altitude $a < 0°$. In this text, we can define **azimuth** as the angle along the horizon to the object measured eastward from true north. However, depending on the convention, azimuth can be defined as being measured from the South instead of the North, or westward instead of eastward. Thus, when doing a coordinates problem that involves the horizontal coordinate system, it is important to pay attention to how azimuth is defined in the problem or declare the convention you are using if one isn't given.

There are a few other important features on the horizontal coordinate system. The **zenith** is the point of 90° altitude, or the highest point in the sky. If you were standing upright, your zenith would be straight above your head. The **zenith distance** is the angle distance from the object to the zenith and the complementary angle of the object's altitude. Opposite of the zenith is the **nadir**, which would be straight beneath your feat. The **celestial meridian** is the great circle that




(a) The latitude and longitude spherical coordinate system. The equator and Greenwich Meridian are in bold. (Image Credit: Ocean Drifters)

(b) The horizontal, or altitude-azimuth, coordinate system. (Image Credit: Wikipedia)

Figure 16.6: Spherical coordinate systems.

passes through true north and the zenith. Stars reach their highest (and lowest) points when they cross the meridian, which is called the culmination or meridian transit of the star. We will expand upon this in the Section 16.3.

The horizontal coordinate system is great for telling us where we need to look for to find a star. For example, if we are told the star Polaris has horizontal coordinates $(A, a) = (0°, 30°)$ (using the convention that azimuth is defined eastward from the true north) for a certain location and time, then we simply need to face north and look $30°$ above the horizon. However, the horizontal coordinate system cannot be used to assign coordinates to stars and other celestial bodies because these coordinates are defined relative to the observer. For example, while an observer standing at the North Pole might see Polaris at her zenith ($a = 90°$), an observer standing at the South Pole would see an entirely different star. In fact, they wouldn't even see any of the same stars! Observers at different positions on Earth would see different stars in different positions. Furthermore, due to Earth's rotation, stars will appear to move in the sky as the night progresses, and so a star's horizontal coordinates are not only location-dependent but also time-dependent. Clearly, we need a different coordinate system if we want to assign coordinates to objects!

**Example 16.2.** One day, Robert the astronomer decides to measure altitude and azimuth of the sun. He knows better than to look directly at the sun, and decides to instead use a 1-meter long stick. He props the stick so that it points straight up, and finds the length of the stick's shadow to be 2 meters. Additionally, using a compass and protractor, he finds that the shadow is pointing $20^\circ$ east of true north. What are the horizontal coordinates of the sun at this instant?

*Solution.* The stick, its shadow, and the sun's rays form the sides and hypotenuse of a right triangle, such that

$$\tan a = \frac{1\,\text{m}}{2\,\text{m}}$$

where $a$ is the angle of the sun's rays above the ground, or the altitude. Therefore, the altitude of the sun $a = \arctan 0.5 = 27^\circ$.

The sun is located opposite to where the stick's shadow is pointing. Since the azimuth of the stick's shadow is $20^\circ$, the azimuth of the sun must be $20^\circ + 180^\circ = 200^\circ$. □

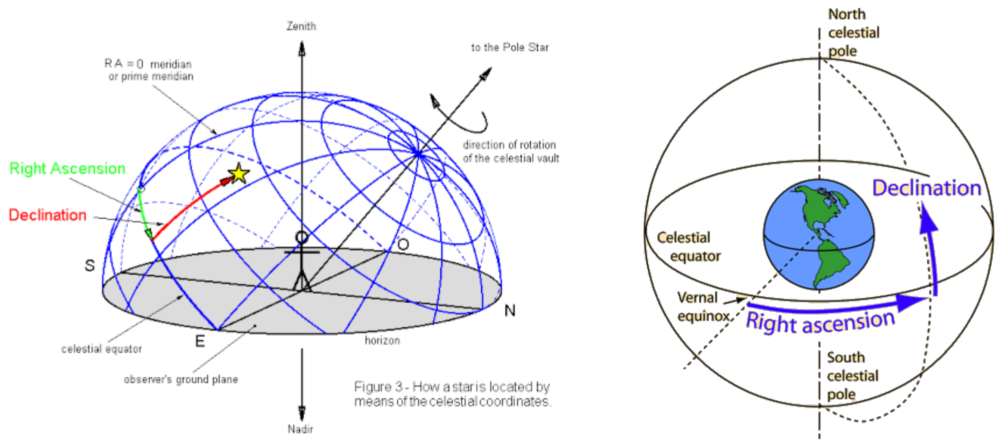
### 16.2.2 The Equatorial Coordinate System

Beginner

In order to define a coordinate system that is fixed with respect to the stars, we must choose reference planes for $0^\circ$ latitude and longitude that do not depend on the observer. There are several reference planes we can choose from, including the celestial equator, ecliptic, and galactic plane, and we will go over each of those in turn.

By far the most useful and commonly used coordinate system is the **equatorial coordinate system**. It uses the celestial equator as the plane of $0^\circ$ latitude. The **celestial equator** is Earth's equator projected onto the celestial sphere; with the celestial equator, we can also define the **North Celestial Pole** and **South Celestial Pole**, which are the north and south poles projected onto the celestial sphere, respectively. Equatorial coordinates are given by declination and right ascension, which are analogous to latitude and longitude, respectively. Declination (Dec, $\delta$) is the angle of the object above the celestial equator; stars in the northern (celestial) hemisphere have positive declination and those in the southern (celestial) hemisphere have negative declination. Right ascension (RA, $\alpha$) is the angle measured eastward along the celestial equator from a reference point to the object. This reference point of $0^\circ$ right ascension is a point on the celestial equator called the vernal equinox, which we will describe in more detail in Section 16.2.3; it is currently located in the constellation Pisces. For now, you may think of it as an arbitrary point on the celestial equator.

It may be confusing to think what it means to measure an angle "eastward" if you are used to thinking about North, South, East, and West as cardinal directions on the ground. Firstly, there's an important distinction between the North/South and East/West directions. If you travel North you will eventually reach the North Pole and begin travelling South if you were to continue in the same direction, while you may travel East or West indefinitely without changing direction. Traveling East simply means that if you were to view your trajectory from above the North Pole, it would be counterclockwise. In fact, it is probably easier to think about East and West like this and just extending this to the celestial sphere: going east along the celestial equator just means that you are going counterclockwise (as viewed from above North Celestial Pole). If you are familiar with the right hand rule, if you curl your fingers in the direction that longitude




(a) The equatorial coordinate system from the point of view of an observer. At non-polar latitudes, the equator and celestial poles will appear at an angle above the horizon. (Image Credit: funsci.com)

(b) The celestial equator, upon which RA and Dec are measured, is the projection of Earth's equator on the celestial sphere. (Image Credit: HyperPhysics)

Figure 16.7: The equatorial coordinate system.

is measured, your thumb should point in the direction of the positive (North) pole of most coordinate systems.

Declination (and most angles in astronomy) are measured using the sexagesimal system of °:':" or `degrees:arcminutes:arcseconds`, with 60 arcminutes in a degree and 60 arcseconds in an arcminute. 1" is thus 1/3600°. Note also that ∼206265" are in a radian: this is useful in small angle calculations. Right ascension, on the other hand, is usually measured in HH:MM:SS (`hours:minutes:seconds`). The conversion is as follows: 24 hours corresponds to 360°, and of course there are 60 minutes in an hour and 60 seconds in a minute. Thus, 1 hour is 15°, 1 minute is 15', and 1 second is 15". The reason for using this system of measuring angles is because it makes tracking the movement of stars easier. This will become more clear in Section 16.3.

**Example 16.3.** What is the angular distance between the two brightest stars in Orion, Betelgeuse (RA: $05^h\ 55^m\ 10.3^s$, Dec: +07° 24' 25.4") and Rigel (RA: $05^h\ 14^m\ 32.3^s$, Dec: -08° 12' 05.9")?

*Solution.* Draw a spherical triangle with the North Celestial Pole and the two stars as vertices:

We would like to find the length of side $RB$. Since this problem involves three sides and an angle that is opposite the unknown side, the spherical law of cosines is perfect for solving this problem.

$$\begin{aligned}\cos RB &= \cos(90^\circ - \delta_R)\cos(90^\circ - \delta_B) + \sin(90^\circ - \delta_R)\sin(90^\circ - \delta_R)\cos(\alpha_B - \alpha_R)\\ &= \sin\delta_R \sin\delta_B + \cos\delta_R \cos\delta_B \cos(\alpha_B - \alpha_R)\end{aligned}$$

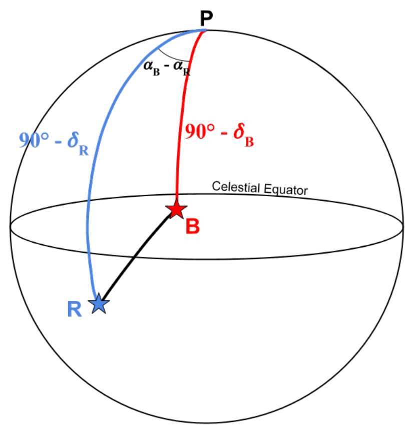



Figure 16.8: A spherical triangle with Betelgeuse (B) and Rigel (R) as seen from outside the celestial sphere. Note that this is not to scale. (Own Work)

We must convert the given coordinates into decimal degrees:

$$\begin{aligned}
\delta_R &= \text{-}08^\circ\ 12'\ 05.9" = -(8^\circ + (12/60)^\circ + (5.9/3600)^\circ) \\
&= -8.2016^\circ \\
\delta_B &= +07^\circ\ 24'\ 25.4" = 7^\circ + (24/60)^\circ + (25.4/3600)^\circ \\
&= 7.4071^\circ \\
\alpha_B - \alpha_R &= 05^h 55^m 10.3^s - 05^h 15^m 32.3^s = 55.1717^m - 15.5383^m \\
&= 39.6333\text{ m} \times \frac{15'}{1\text{ m}} = 594.5' \times \frac{1^\circ}{60'} \\
&= 9.9083^\circ
\end{aligned}$$

Using these numbers, we have

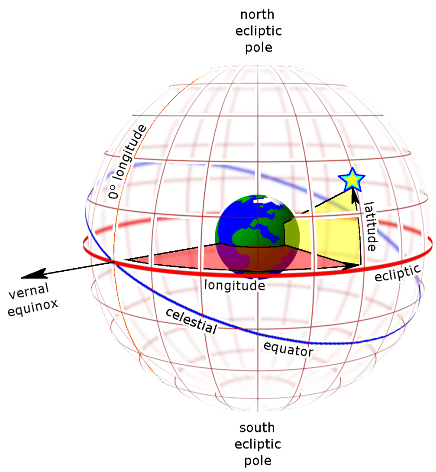
$$\begin{aligned}
\cos RB &= 0.9485 \\
RB &= 18.4714^\circ
\end{aligned}$$

□

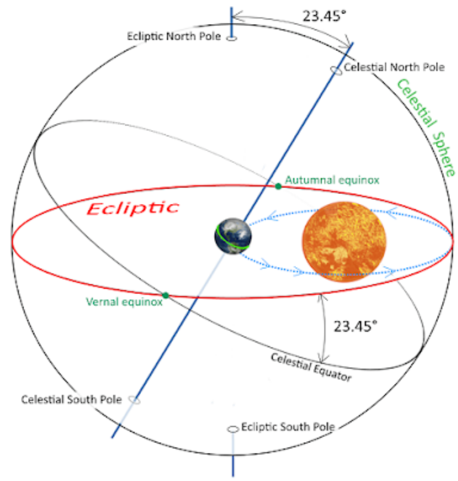
### 16.2.3 The Ecliptic Coordinate System

Beginner

Another celestial coordinate system is the **ecliptic coordinate system**, which uses the **ecliptic** great circle as its reference plane (see Figure 16.9). The ecliptic is the plane of Earth's orbit and the solar system; the sun and planets are all located on the ecliptic. The **Ecliptic North Pole** and **Ecliptic South Pole** are then the points on the sphere orthogonal to the ecliptic. The ecliptic coordinate system uses ecliptic longitude ($\lambda$ or $l$) and ecliptic latitude ($\beta$ or $b$). Similar to the equatorial coordinate system, ecliptic longitude is measured eastward from the vernal equinox.




(a) How ecliptic longitude and latitude are measured. (Image Credit: Wikipedia)




(b) The ecliptic is the plane of Earth's orbit; the equator is the plane of Earth's rotation. (Image Credit: Unknown)

Figure 16.9: The ecliptic coordinate system. Notice how the ecliptic plane and equatorial plane are offset (by 23.45°); the vernal equinox is one of their intersections.

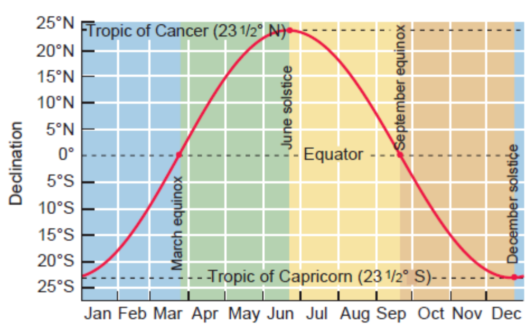
Note that the ecliptic coordinate system can be either geocentric or heliocentric, depending on the application. Heliocentric ecliptic coordinates are useful for tracking solar system objects that orbit around the sun. This is different from the equatorial coordinate system, which is always geocentric.

With the introduction of the ecliptic, we can now define the vernal equinox. You probably know that Earth's rotation axis is tilted by an angle, and that this tilt is the reason for Earth's seasons. More precisely, Earth's equatorial plane is tilted with respect to the ecliptic (alternatively, Earth's rotation axis is tilted with respect to the ecliptic poles). This inclination angle is called the **axial tilt** or the **obliquity of the ecliptic**, usually donated by epsilon. It is currently equal to 23.45°. Intermediate

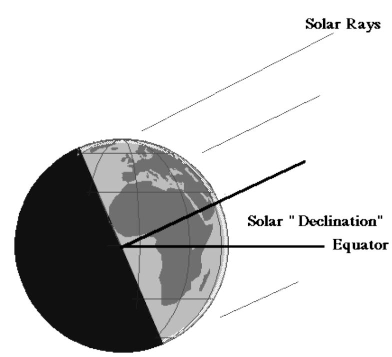
Because the two planes are inclined with respect to each other, they intersect at two points (see Figure 16.9): the **vernal equinox** and the **autumnal equinox**. This gives us one step in defining the vernal equinox. But how do we distinguish between these two intersection points? If we trace along the ecliptic, one becomes an ascending node and the other a descending node with respect to the celestial equator. For example, if we trace along the ecliptic counterclockwise as viewed from the Ecliptic North Pole (eastward), at the vernal equinox we pass from below the celestial equator to above the celestial equator, and at the autumnal equinox we pass from above the celestial equator to below. We can then describe the vernal equinox as the ascending node of the ecliptic with respect to the celestial equator, travelling eastward.

This description is rather wordy and convoluted. An easier way to describe this is just to follow the movement of an object along the ecliptic: the sun! As Earth orbits around the sun, the sun will appear to move with respect to the celestial sphere along the ecliptic (see Figure 16.11). Since the Earth orbits the sun counterclockwise (as viewed from the Ecliptic North Pole), the sun appears to us to travel east along the ecliptic, increasing in not only right ascension but also declination throughout the year.

As the sun travels around the ecliptic, it will cross the equator at the vernal equinox, reach a point of maximum declination of $+23.45°$, cross the equator again at the autumnal equinox, reach a point of minimum declination of $-23.45°$, and reach the vernal equinox again exactly one year later. Recall that declination is the angle of an object above or below the celestial equator. We can see this in a plot of the sun's declination throughout the year (see Figure 16.10(a)). The point of maximum declination is called the summer solstice and marks the first day of summer. At this point, the sun's rays come in at an angle above the equator, and thus the Northern Hemisphere has its summer (see Figure 16.10(b)). Likewise, the winter solstice marks the first day of winter, the autumnal equinox the first day of autumn, and the vernal equinox the first day of spring. Thus, *the vernal equinox can be described as the position of the sun on the celestial sphere on the first day of spring, the vernal or spring equinox!*.




(a) The declination of the sun throughout the year: it is maximum at the summer solstice, minimum at the winter solstice, and zero at the equinoxes. (Image Credit: All Geography Now)




(b) At the summer solstice, the Northern Hemisphere is tilted towards the sun. Solar rays come in at an angle with respect to the equator, giving the sun a positive declination. (Image credit: ESO)

Figure 16.10: The sun varies in equatorial coordinates as it travels along the ecliptic.

Beginner

The band of constellations on the ecliptic is called the zodiac. Thus, most solar system objects will be within a zodiac constellation. There are 12 traditional zodiac constellations, but there are actually 13 on the ecliptic (Ophiuchus being the odd one out). The sun will pass through the various zodiac constellations throughout the year, and this is the basis for different birth dates corresponding to different zodiac constellations in astrology. However, this system was established over 2,000 years ago in ancient Greece (largely by Ptolemy), and it no longer has any astronomical significance due to a phenomenon called the precession of the equinoxes (see

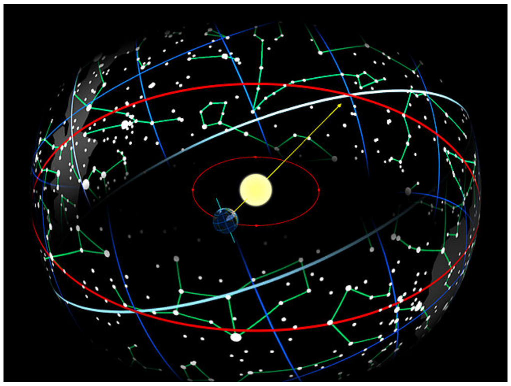


Figure 16.11: The sun appears to travels along the ecliptic throughout the year due to Earth's orbit, passing through various zodiac constellations and changing in right ascension and declination. (Image Credit: EarthSky)

Section 16.2.5).

**Example 16.4.** What are the approximate equatorial and ecliptic coordinates for the sun at the a) vernal equinox, b) autumnal equinox, c) summer solstice, and d) winter solstice?

*Solution.* Let's begin with the sun's ecliptic coordinates. The sun always stays on the ecliptic plane as Earth orbits, so its ecliptic latitude is always $\beta = 0°$. The Earth moves counterclockwise in its orbit (as viewed from above the North Pole), and thus the sun appears to move eastward relative to Earth.

The sun is at the vernal equinox on the vernal equinox (by definition), so it has ecliptic longitude $\lambda = 0°$. For an (approximately) circular orbit, the sun appears to travel with a constant angular speed. Thus, it has ecliptic longitude $\lambda = 90°$ at the summer solstice, $\lambda = 180°$ at the autumnal equinox, and $\lambda = 270°$ at the winter solstice.

Knowing this, what are the sun's equatorial coordinates at these points? The celestial equator and ecliptic intersect at the vernal and autumnal equinoxes. The vernal equinox has equatorial coordinates $(\alpha, \delta) = (0^h, 0°)$, and the autumnal equinox is opposite of the vernal equinox on the celestial sphere, with equatorial coordinates $(\alpha, \delta) = (12^h, 0°)$. Due to symmetry, it is not unreasonable to assume the summer and winter solstices to be exactly halfway between the equinoxes. (We will prove this rigorously in Section 16.4.) The celestial equator and ecliptic are offset by $\epsilon = 23.45°$, and the sun's declination reaches a maximum at the summer solstice and a minimum at the winter solstice. Thus, the sun's equatorial coordinates at the summer solstice is $(6^h, +23.45°)$, and the sun's equatorial coordinates at the winter solstice is $(18^h, -23.45°)$.

□

### 16.2.4 The Galactic Coordinate System

Beginner

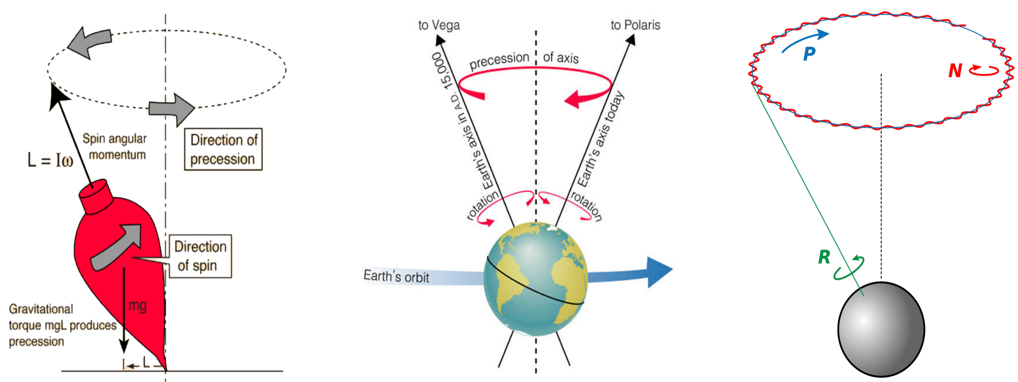
Finally, we will discuss briefly the galactic coordinate system, composed of galactic latitude ($b$) and galactic longitude ($l$), used often in applications involving surveying or mapping the Milky Way. As you might guess, the galactic coordinate system uses the Milky Way as the plane of 0 galactic latitude. Galactic longitude is measured eastward along the Milky Way from the galactic center. Because the Milky Way's disk has a thickness, there are stars all around us in our local neighborhood. However, there is an increased concentration of stars near 0° galactic latitude as well as towards the center of the galaxy (0° galactic longitude) simply because most stars in the Milky Way are located within the disk.

Note that the galactic coordinate system is heliocentric, not geocentric.

### 16.2.5 Shifting Coordinate Systems

Beginner

There is one major problem for the equatorial coordinate system: Earth's rotational axis is slowly shifting! This phenomenon is called **axial precession**. Precession is the movement of a rotating body's rotational axis, and can be commonly seen in spinning tops and gyroscopes. Earth undergoes a similar motion, precessing in a 23.45° cone about the ecliptic poles in a 26,000 year cycle.




(a) The precession of a top. (Image Credit: HyperPhysics)

(b) Earth's precession about the ecliptic poles. (Image Credit: Sara's Astronomy Blog)

(c) Schematic with rotation, precession, and nutation. (Image Credit: Wikipedia)

Figure 16.12: How precession works.

The position of the North Celestial Pole traces out a circle on the celestial sphere: the star Thuban ($\alpha$ *Dra*) was the North Star around 3000 BCE, and in around 12,000 years the North Celestial Pole will be quite close to Vega (see Figure 16.13). Polaris was not always and will not always be the North Star! If the poles are shifting, so are the celestial equator and equinoxes; the equinoxes will slowly shift over time along the ecliptic in a phenomenon called the precession of the equinoxes.

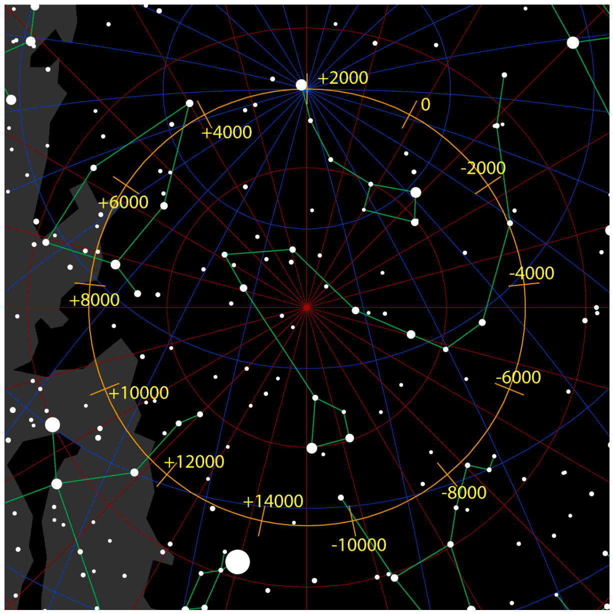



Figure 16.13: Earth's North Celestial Pole traces out a 23.45° circle about the North Ecliptic Pole. While Polaris is the North Star now, it won't be in a few thousand years. (Image Credit: Wikipedia)

Furthermore, Earth's obliquity also varies over time. Earth "wobbles" in its precession, causing changes in obliquity with an amplitude of 9.2" and period of 18.6 years (see Figure **??**): this is primarily caused by tidal effects of oscillations in the moon's orbital plane. Additionally, there are various other effects that contribute to complex long-term variations of Earth's obliquity. This phenomenon is called **nutation**. Clearly, Earth's celestial poles and equator are shifting over time with respect to the stars.

To fix this issue, astronomers will report the coordinates of an object based on the celestial equator and vernal equinox of a specific time and date; this is called the **epoch**. We currently use the astronomical epoch J2000, which is defined at noon UTC of January 1, 2000 (JD = 2451545.0). Thus, the real celestial equator is slightly offset from 0 declination, since the celestial equator has shifted since January 2000. Astronomers typically change reference epochs in intervals of 50 years; reference dates of January 1, 1950 and January 1, 1900 were used in the past.

The ecliptic has similar issues (Earth's orbit also precesses due to effects of general relativity), but Earth's orbital plane is stable on a much longer timescale than Earth's rotational axis, so they can largely be ignored. Nonetheless, specifying the epoch is customary when giving coordinates in any coordinate system.

## 16.3 The Rotating, Orbiting Earth

### 16.3.1 The Stars' Diurnal Motion

Intermediate

The reason why the equatorial coordinate system is especially useful is because the Earth rotates, and the poles of the equatorial system are aligned with Earth's spin axis (by definition). While it is Earth that is physically rotating, from our perspective it appears to be the celestial sphere that is rotating around us. This is demonstrated most clearly by time-lapse photographs of the night sky (see Figure); stars appear to rotate about Polaris (the North Celestial Pole) in the Northern Hemisphere. This is called the stars' **diurnal motion**. Since Earth rotates eastward (the sun rises in New York before it rises in California), the celestial sphere and its constituents (including the sun) will appear, from our point of view, to rotate westward, rising from the east and setting towards the west. Stars will rotate counterclockwise around the North Celestial Pole in the Northern Hemisphere, and clockwise around the South Celestial Pole in the Southern Hemisphere[1]. Note that unless the star is on the celestial equator, they all travel on small circles (as opposed to great circles).

Because the equatorial coordinate system is defined this way, all objects with the same right ascension will **culminate**, or cross the meridian, at the same time. Additionally, as time passes and the celestial sphere rotates westward, objects of greater right ascension begin transiting. For example, let's say Cygnus ($\alpha \approx 21^h$) is transiting right now. 4 hours later, Earth has rotated an angle of $4h$ to the East, and thus a constellation of $21^h + 4^h = 1^h$ is now transiting (Andromeda). This is why right ascension uses HH:MM:SS rather than °:':".

Since stars appear to rotate around the North Celestial Pole, the North Celestial Pole itself does not move throughout the night (See Figure 16.14), and thus the position of the North Celestial Pole on the night sky only depends on the observer's location on Earth. It is obvious that the North Celestial Pole will always be towards the North ($A = 0°$ in horizontal coordinates), but what is its altitude? We might imagine that the North Celestial Pole would be directly overhead (at the zenith) at the North Pole, lower in the sky near the equator, and not visible in the Southern Hemisphere.

This is entirely correct, and in fact *the altitude of the North Celestial Pole is simply the observer's geographic latitude* (and because the South Celestial Pole is opposite of the North Celestial Pole, its altitude is equal to the opposite of the observer's latitude):

$$a_{NCP} = \phi \qquad\qquad a_{SCP} = -\phi \tag{16.1}$$

(Note that this assumes $\phi < 0°$ for Southern latitudes, i.e. $\phi = 30°$ S would be written as $\phi = -30°$.)

[1] Why are stars moving counterclockwise in the Northern Hemisphere? Remember that the celestial sphere rotates westwards, which means if we were to view the stars' trajectories from outside the celestial sphere above the North Celestial Pole, we would see clockwise paths. However, since we are viewing the celestial sphere from the inside, westward movement on the celestial sphere corresponds to counterclockwise movement in the Northern Hemisphere. The same logic applies to the Southern Hemisphere.


Figure 16.14: Stars' diurnal motion throughout the night about the North Celestial Pole (Polaris) is captured in this time-lapse photo. The meridian circle is marked in red; stars crossing the meridian are reaching their highest (upper culmination) or lowest (lower culmination) points. Usually, culmination refers to upper culmination. (Adapted from Unknown)

At the North Pole ($\phi = 90°$) the altitude of the North Celestial Pole is 90°, at the equator the altitude of the North Celestial Pole is 0° (just on the horizon), and at the South Pole the altitude of the North Pole is -90° (at nadir). The South Celestial Pole is always opposite of the North Celestial Pole, so if the North Celestial Pole has an altitude of 30° (observer is at latitude $\phi = +30°$), the South Celestial Pole must have an altitude of -30°. A proof is given in Figure 16.17.

This result allows us to pinpoint the North Celestial Pole as well as the celestial equator in the night sky, and will be useful later when we derive the conversion between equatorial and horizontal coordinates. It also tells us the condition for **circumpolar** stars, which are stars that never dip below the horizon in their diurnal paths. For example, if you are at latitude +30 N, then the North Celestial Pole is 30° above the horizon. Then, stars that are within 30° of the North Celestial Pole (i.e. $\delta > 60°$) will have circular paths that are completely above the horizon. The general condition for circumpolar stars is:

$$\begin{aligned} &\delta > +90° - |\phi| && \text{for the Northern Hemisphere} \\ &\delta < -90° + |\phi| && \text{for the Southern Hemisphere} \end{aligned} \tag{16.2}$$

## 16.3.2 Sidereal Time

How long does it take each star to rotate through 360°? Since this is the same as asking how long it takes for the Earth rotate 360°, one might conclude that it would take 1 day, or 24 hours.

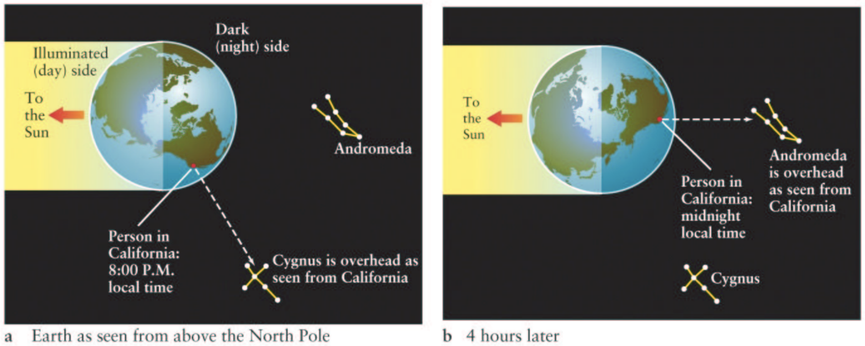



Figure 16.15: Due to Earth's rotation, different stars and constellations will cross the Meridian as time passes. (Image Credit: Unknown)

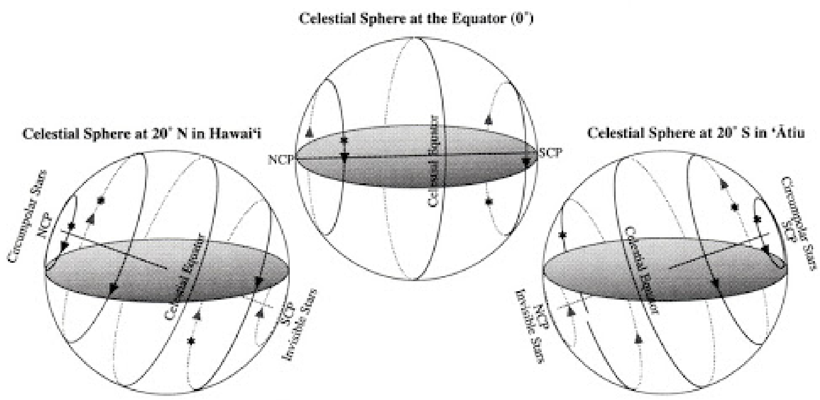



Figure 16.16: Stars' diurnal motion about the North Celestial Pole for observers at different latitudes. (Image Credit: hokulea.com)

The real answer is, however, slightly less than 24 hours; to be more precise, it is 23 hours, 56 minutes, and 4.1 seconds. This discrepancy lies in the difference between the **solar day** and the **sidereal day**.

A sidereal day is equal to Earth's rotation period–the time it takes for Earth to rotate 360°. We may also think of it as the amount of time between two culminations of the same star. In general, **sidereal time** is time with respect to the stars and distant objects. Therefore, it is 4 *sidereal* hours that corresponds to 4 hours of right ascension, and 24 *sidereal* hours (or one sidereal day) passes before the same star rises again.

The time system that we use, however, is **mean solar time**, or time with respect to the sun. You may define one solar day as the period of time between two consecutive noons or meridian transits of the sun. We use solar time for obvious reasons: we want day and night to correspond to consistent times! Sidereal and solar time are different because Earth orbits

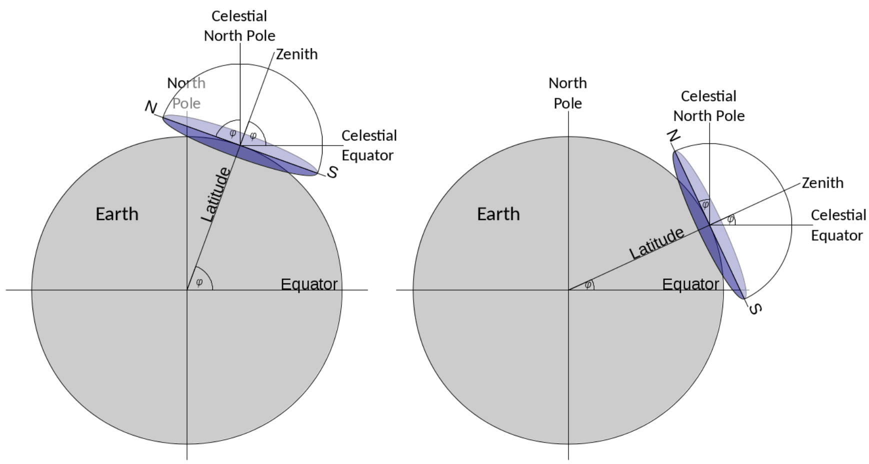



Figure 16.17: Proof that the altitude of the North Celestial Pole is equal to the observer's latitude, assuming a spherical Earth. The same analysis applies to the South Celestial Pole. (Image Credit: M. Nielbock, T. Muller)

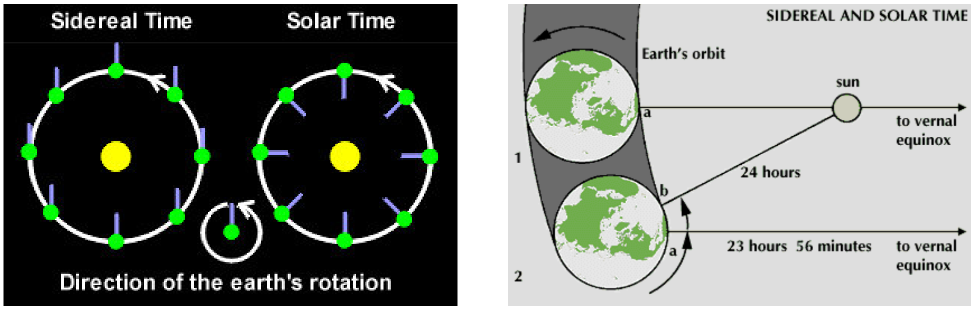
around the sun. As Earth moves around in its orbit, at 12 PM each day there are different stars on the meridian (see Figures 16.11, 16.18); recall that the sun moves through various zodiac constellations throughout the year. Because Earth orbits in the same direction as its rotation (counterclockwise as viewed from the North), Earth must rotate slightly more than 360° between two consecutive noons. Thus, the solar day is slightly longer than the sidereal day by about 4 minutes; stars rise about 4 minutes earlier each day. In one complete orbit around the sun, there is one more sidereal day than solar day. We can derive this number:

**Example 16.5.** How much longer is a mean solar day than a sidereal day?

*Solution.* Let's define one solar day, equal to 24 (solar) hours, as the time between two consecutive noons. There are approximately 365.256 solar days in a sidereal year[2]. Assuming a circular orbit, after one solar day, Earth moves $\theta = \frac{360^\circ}{365.256 \text{ days}} = 0.985610^\circ$ in its orbit; therefore, Earth must rotate 360.985610° in order to face the sun again. If one solar day is the time it takes to rotate 360.985610°, then one sidereal day, the time it takes to rotate 360°, must be

$$\begin{aligned} 24 \text{ hours} \times \frac{360^\circ}{360.985610^\circ} &= 23.9345 \text{ hours} \\ &= 23 \text{ hours } 56 \text{ minutes } 4.1 \text{ seconds} \end{aligned}$$

□




(a) As Earth moves around the sun, there are different stars on the meridian at each noon. Sidereal time and solar time do not match. (Image Credit: Tim Foster)

(b) Earth has to rotate slightly more than 360° in order to face the sun again. Thus, the solar day is slightly longer than the sidereal day. (Image Credit: Britannica)

Figure 16.18: Sidereal time.

LST (**local sidereal time**) is standardized by defining $LST = 0^h$ to be when the vernal equinox (and any objects of $\alpha = 0^h$) crosses the meridian. Thus, every day at 0:00 local sidereal time (which corresponds to different solar times throughout the year), stars of $\alpha = 0^h$ are transiting. Similarly, one (sidereal) hour later at local sidereal time 01:00, stars of $\alpha = 1^h$ are overhead, stars of $\alpha = 0^h$ are one hour past their upper culmination, and stars of $\alpha = 0^h 30^m$ are 30 minutes past their upper culmination. This is usually expressed as the hour angle ($H$), which you may think of as the sidereal time that has passed since an object's most recent upper culmination. The local sidereal time is often described as the *hour angle of the vernal equinox*, or the *right ascension of the stars on the meridian*. Therefore, we have the following relation between the right ascension ($\alpha$) and hour angle ($H$) of an object and the local sidereal time ($LST$):

$$H = LST - \alpha \tag{16.3}$$

Formally, hour angle is defined as the angle between the plane containing the North Celestial Pole and the zenith (the meridian plane) and the plane containing the North Celestial Pole and the object, or the "hour circle". Hour angle is measured westward, the same direction as the rotation of the stars. A diagram of what this looks like is found in Figure **??**. An hour angle of 0 means that the object is on the meridian, and an hour angle of $15° = 1$ hour means that the object has rotated 15° past the meridian. Still, I have always found it most convenient to think about hour angle as the sidereal time that has passed since the object's last meridian transit.

Finally, note that LST is *local* sidereal time, for the same reason that we have time zones. Stars will rise at different times at different longitudes. For example, if City B has a longitude of

[2] The sidereal year is the time it takes for Earth to make one full revolution or 360°; this is the number we are looking for. On the other hand, the tropical year is the time it takes for the sun's ecliptic longitude to increase 360°, or the time between two consecutive spring equinoxes (or summer solstices, etc.). Due to the precession of the equinoxes, the sidereal year is longer by about 20 minutes.

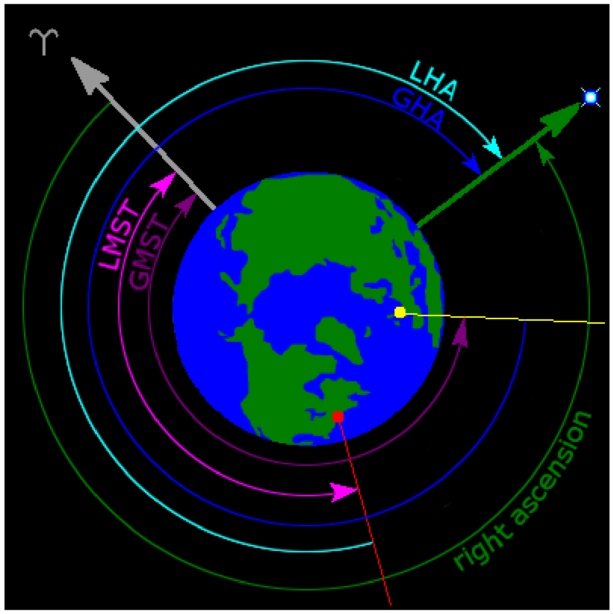



Figure 16.19: The relation between hour angle (LHA), local sidereal time (LMST), Greenwich hour angle (GHA), Greenwich sidereal time (GMST), and right ascension for the star marked by the green arrow. The vernal equinox $\gamma$ is given by the gray arrow. The Greenwich Meridian and the observer's location are given by the yellow and red dots, respectively. (Image Credit: Wikipedia)

30° E and City A has a longitude of 0°, City B will see the same star rise 30° = 2 hours earlier. Many astronomical tables use Greenwich Mean Sidereal Time (GMST), the local sidereal time on the Greenwich meridian, rather than Local Sidereal Time, since the latter depends on the observer's longitude. The two are related with

$$LST = GMST + l \tag{16.4}$$

Of course, you must convert your longitude into HH:MM:SS before using this equation. (Clearly, measuring angles with units of time is really useful!). A diagram relating all of these quantities is shown in Figure 16.19.

**Example 16.6.** Approximately when are local sidereal time and local mean solar time equal?

*Solution.* On the day of the vernal equinox, the sun is on the vernal equinox. At noon (local mean solar time is 12 PM), the sun is transiting. The local sidereal time is the hour angle of the vernal equinox; since the sun is on the vernal equinox, the local sidereal time is $LST = 0^h$.

Local sidereal time and local mean solar time are 12 hours apart on the spring equinox, and their difference increases by 24 hours in one year. Therefore, local sidereal time and local mean solar time are the same half a year after the vernal equinox, at the autumnal equinox. □

### 16.3.3 Coordinate System Conversions

Advanced

We now have all the necessary tools in order to derive the formulas for converting between spherical coordinate systems. The methods introduced in this section can be used to solve a large proportion of USAAAO and IOAA celestial coordinates problems, and any Science Olympiad Astronomy problems that involves celestial coordinates.

Converting between horizontal and equatorial coordinates is extremely useful. Equatorial coordinates are by far the most commonly used celestial coordinate system, and in order to point our telescopes on that object we must know its horizontal coordinates. Conversely, if we spot a previously undiscovered object in the sky, we want to be able pinpoint its equatorial coordinates.

For any arbitrary object in the sky, we can measure its altitude $a$ and azimuth $A$ (Figure 16.1(a)). Here, we are using the convention that azimuth is measured eastward from true north. The cardinal directions are marked on the horizon, as is the location of the observer $O$ and the zenith $Z$.

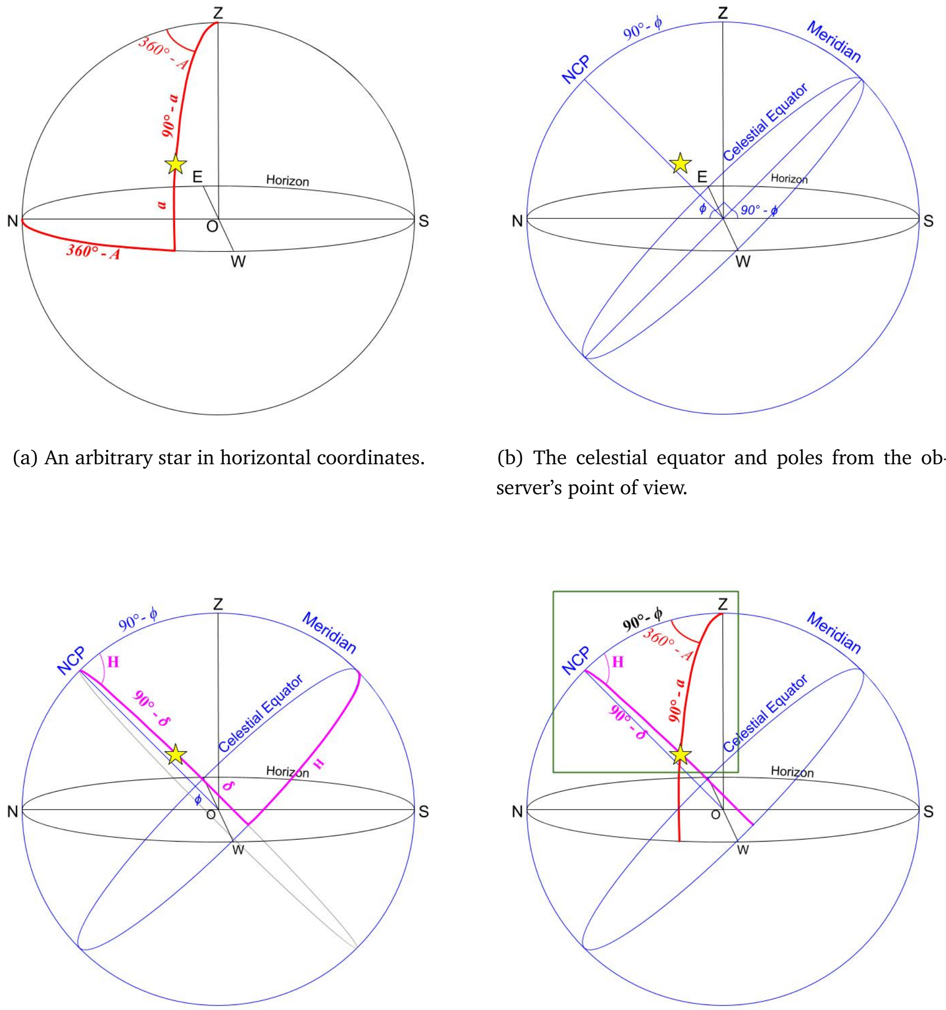
What about its equatorial coordinates? We can draw the North Celestial Pole and celestial equator for an observer's local sky (Figure 16.1(b)); recall that the North Celestial Pole points towards true north and that its altitude is equal to the observer's latitude. Since the zenith is orthogonal to the horizon, the angle between the zenith and the North Celestial Pole is $90° - \phi$. The celestial equator is orthogonal to the poles, so it is tilted $90° - \phi$ from the horizon; it intersects with the horizon at true east and true west. Note that this assumes the observer is in the Northern Hemisphere; in the Southern Hemisphere, the South Celestial Pole would be visible with an altitude equal to the absolute value of the latitude, pointing towards true south.

Next, we draw in the object's equatorial coordinates (Figure 16.1(c)). Instead of right ascension, we are using the object's hour angle, since we already know the two are related with Equation 16.3 ($H = LST - \alpha$). The object's hour circle is shown in gray; the hour angle is the angle between the meridian and the hour circle. Hour angle $H$ is measured westward from the meridian plane to the meridian plane, and declination $\delta$ is measured as the angle north of the celestial equator.

Finally, combining Figures 16.20(a) and 16.20(c), we can obtain a spherical triangle that relates an object's horizontal and equatorial coordinates in Figure 16.20(d). I like to call this the "holy triangle" (or perhaps more appropriately, "pole-y triangle") of celestial coordinates, since it is used to solve so many celestial coordinates problems. It consists of a spherical triangle with two coordinate poles and the object as vertices.

Now, we need only apply the spherical laws of cosines and sines to this spherical triangle to obtain the quantities we need. We assume that the local sidereal time and the geographic location of observation is known.

To convert from equatorial coordinates $(\alpha, \delta)$ to horizontal coordinates $(a,A)$, we first find the hour angle via Equation 16.3.




(a) An arbitrary star in horizontal coordinates.

(b) The celestial equator and poles from the observer's point of view.

(c) The equatorial coordinates of an arbitrary star from the observer's point of view.

(d) The final spherical triangle.

Figure 16.20: Drawing the coordinate conversion spherical triangle and associated angles. (Own Work)

$$H = LST - \alpha$$

We then apply the spherical law of cosines (Theorem 16.1) to find the altitude $a$:

$$\begin{aligned} \cos(90^\circ - a) &= \cos(90^\circ - \delta)\cos(90^\circ - \phi) + \sin(90^\circ - \delta)\sin(90^\circ - \phi)\cos H \\ \sin a &= \sin\delta\sin\phi + \cos\delta\cos\phi\cos H \end{aligned} \tag{16.5}$$

There is no ambiguity for $a$ when using arcsin, since the altitude only varies from $-90°$ to $90°$ anyway.

Next, we can find the azimuth $A$ from the spherical law of sines (Theorem 16.2) (see Example 16.1.):

$$\sin A = -\sin H \frac{\cos\delta}{\cos a} \tag{16.6}$$

Alternatively, we can use the spherical law of cosines:

$$\cos A = \frac{\sin\delta - \sin a \sin\phi}{\cos a \cos\phi} \tag{16.7}$$

Note that there is ambiguity when using arcsin or arccos because the azimuth can range from $0°$ to $360°$. Often you can tell which one it should be, but solving for both $\sin A$ and $\cos A$ is necessary to be systematic and precise.

We can use a similar procedure to convert from horizontal to equatorial coordinates. First, use the spherical law of cosines to solve for the declination $\delta$:

$$\sin\delta = \sin\phi \sin a + \cos\phi \cos a \cos A \tag{16.8}$$

Then, use the spherical law of sines and/or cosines to solve for the hour angle $H$:

$$\sin H = -\sin A \, \frac{\cos a}{\cos\delta} \tag{16.9}$$

$$\cos H = \frac{\sin a - \sin\delta \sin\phi}{\cos\delta \cos\phi} \tag{16.10}$$

Finally, find the right ascension $\alpha$ by applying Equation 16.3:

$$\alpha = LST - H$$

I advise against just memorizing these conversion formulas for several reasons. Firstly, many problems are more complex (or less complex!) than simply converting from one coordinate system to another, and it is important to understand how to draw the right triangle and apply the right spherical trigonometric formulas. Secondly, this derivation is specific to a Northern Hemisphere observer measuring azimuth eastward from true north. Thirdly, memorizing formulas won't do you any good where you are supposed to show your work. Lastly, this method – drawing a spherical triangle with two poles and the object as vertices and applying the right spherical trigonometric formulas – can be used to convert between other coordinate systems as well! We will derive a portion of the conversion between equatorial and ecliptic coordinates in Section 16.4, and the rest will be left as an exercise to the reader.

**Example 16.7.** On his way to a conference, Robert the astronomer crash landed on an island with nothing but a watch (set to GMT), a calculator, an astronomical almanac, and a very precise sextant, which can measure the angles of stars and other objects above the horizon. He observes Vega (which has coordinates $(\alpha, \delta) = (18^h\,36^m\,56^s, +38°\,47'\,1")$, according to his astronomical almanac) rising with an altitude of $65.628°$. He also observes Polaris with an altitude of $33.736°$ above the horizon. He checks his watch: it is 7:48 PM GMT on March 21, the vernal equinox. Is it possible for Robert to figure out his location from this information? If so, where did he crash land?

*Solution.* The altitude of Polaris immediately gives us Robert's latitude $\phi = 33.736°$. (We assume here that Polaris is exactly on the North Celestial Pole). To calculate his longitude, we can compare his local time to his watch's time.

Our known quantities are $\alpha$, $\delta$, $a$, and $\phi$. Using those to solve for the hour angle $H$ can give us the local sidereal time. If we draw the appropriate spherical triangle (Figure 16.20(d)), we can use the spherical law of cosines to solve for $H$, yielding the spherical coordinate conversion Equation 16.10. Converting Vega's equatorial coordinates to decimal degrees, we have $\delta = 38.7836°$ and $\alpha = 279.2333°$.

$$\begin{aligned}\cos H &= \frac{\sin a - \sin\delta\sin\phi}{\cos\delta\cos\phi} \\ &= \frac{\sin 65.628° - \sin 38.7836° \sin 33.736°}{\cos 38.7836° \cos 33.736°} = 0.86851\end{aligned}$$

Because Vega is rising, the hour angle must be negative. Thus,

$$H = -\arccos 0.86851 = -29.7137°.$$

Next, we can find the local sidereal time with Equation 16.3,

$$\begin{aligned}LST &= H + \alpha \\ &= -29.7137° + 279.2333° = 249.5196°\end{aligned}$$

On the vernal equinox, the local mean solar time is offset from the local sidereal time by 12 hours (see Example 16.6). GMT is equal to the local mean solar time at the Greenwich meridian, which in this case is 7:48 PM $= 19.8^h$. Thus, the local sidereal time at the Greenwich meridian (GMST) is $19.8^h - 12^h = 7.8^h$. Converting to degrees, we have $GMST = 117°$.

Finally, using Equation 16.4 to find the longitude:

$$l = LST - GMST = 132.5°$$

It looks like Robert has crash landed at 33.7°N, 132.5°E, the "Cat Island" Aoshima in Japan!

□

## 16.4 Time Systems and the Equation of Time

In the previous section we introduced the concept of solar time vs sidereal time. Unfortunately, we made two key assumptions that are not entirely accurate: Advanced

1. The Earth is in a perfectly circular orbit.

2. Earth's orbital plane (ecliptic) and rotational plane (celestial equator) are aligned.

We assumed 1) when we assumed that Earth travels at a constant angular speed throughout its orbit. We assumed 2) when we simply added 0.98561° to 360° when in reality these angles are not in the same plane. (The Earth rotated 360° in the equatorial plane; the Earth travelled in its orbit 0.98563° in the ecliptic plane.) Because of these two effects, the length of time between two consecutive noons (meridian transits of the sun) varies throughout the year. It would be pretty inconvenient if 24 hours referred to different lengths of time depending on where Earth was in its orbit, so the time system that clocks use is **mean solar time**, which is based upon the position of the **mean sun**, which is an imaginary sun that does follow our two assumptions. A mean solar day is the length of time between two consecutive meridian transits of the mean sun, and is constant throughout the year. It is this quantity that is equal to 24 hours. Using the mean sun, all of our analyses in the previous section are still valid. On the other hand, **apparent solar time**, is based upon the real position of the sun. It is the time that a sundial reads, and it is given by the hour angle of the sun (+ 12 hours, since we want $t = 12h$ to be noon).

The discrepancy between mean solar time and apparent solar time is given by the **equation of time**. The equation of time has been used since the 18th century to convert between the time given by a sundial and the time given by mechanical watches. In this text we will consider the equation of time to be the apparent solar time minus the mean solar time ($EoT \equiv t_a - t_m$), but some other texts may consider the opposite.

The equation of time results from superposition of the two effects introduced above. The effect of Earth's eccentric orbit is somewhat more intuitive. When Earth travels in an elliptic orbit, it has a faster angular speed near perihelion (∼January 4) and a slower angular speed near aphelion (∼July 4). Around perihelion, since Earth travels faster around the sun, it must rotate a greater angle to face the sun again, and thus the apparent solar day is longer than the mean solar day. This means the equation of time decreases as apparent solar time falls further and further behind mean solar time. Conversely, around aphelion, the apparent solar day is shorter than the mean solar day, and the equation of time increases. This results in a sinusoidal shape for the equation of time, with minima and maxima at the halfway points between perihelion and aphelion (see Figure 16.22, red). A rigorous mathematical derivation of this effect is beyond the scope of this chapter.

The other contribution to the equation of time is the Earth's obliquity, whose effect is far less obvious. The apparent (real) sun travels on the ecliptic while the mean sun travels on the celestial equator, and due to Earth's axial tilt, these two planes are not the same. (To reiterate, it is obviously not the sun that is orbiting eastward on the ecliptic plane, but rather the Earth. However, it is convenient to think about it as the sun moving along the ecliptic on our geocentric celestial sphere.)

Since we account for the effect of Earth's eccentricity elsewhere, we can assume a circular orbit here to isolate the effect of Earth's obliquity. Then, the apparent sun's *ecliptic longitude*, not right ascension, would increase uniformly throughout the year! On the other hand, since the mean sun travels on the celestial equator, its *right ascension* would increase uniformly throughout the year. The equation of time is then given by the difference in right ascension between the mean sun and apparent sun

$$EoT = t_a - t_m = H_a - H_m = \alpha_m - \alpha_a \tag{16.11}$$

where $H$ is the hour angle and $\alpha$ is the right ascension for the mean and apparent suns; we obtain this from Equation 16.3.

Applying the coordinate conversion methods from Section 16.3.3, we can actually quantify this mathematically. Let the mean sun's right ascension be $\alpha_m(T) = \omega(T - T_g)$, and the apparent sun's ecliptic longitude be $\lambda_a(T) = \omega(T - T_g)$, where $T - T_g$ is the time of year with respect to the vernal equinox and $\omega$ is Earth's angular orbital speed ($= 360°/1$ year). In order to find the equation of time $EoT(T) = \alpha_m(T) - \alpha_a(T)$, we must find the right ascension of the apparent sun $\alpha_a(T)$; we can do this with a coordinate conversion from ecliptic coordinates $(\lambda, \beta) = (\omega(T - T_g), 0)$ to equatorial coordinates!

To draw the appropriate spherical triangle (Figure 16.21), we first draw the celestial equator and ecliptic and their respective poles. The two planes are tilted by $\epsilon = 23.45$, and they intersect at the equinoxes, where the vernal equinox is the ascending node tracing along the ecliptic eastward. Right ascension and ecliptic longitude are measured eastward from the vernal equinox. The sun travels along the ecliptic with an ecliptic latitude $\beta = 0°$.

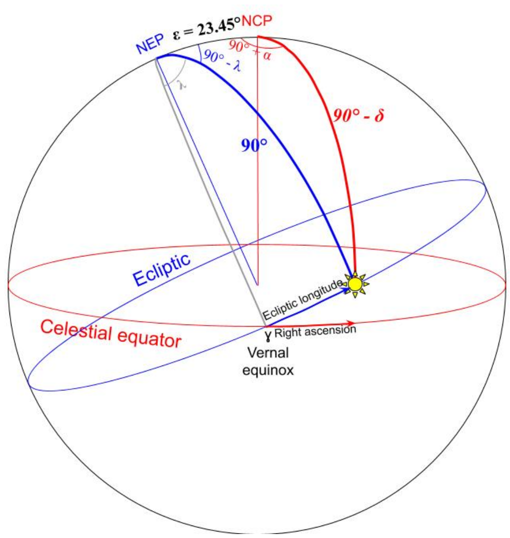



Figure 16.21: The ecliptic (blue) to equatorial (red) coordinate conversion spherical triangles. (Own Work)

Our quantity of interest is the right ascension $\alpha$; the obliquity $\epsilon = 23.45°$, ecliptic longitude $\lambda = \omega(T - T_g)$, and ecliptic latitude $\beta = 0°$ are known. With four consecutive sides and angles, we can apply the cotangent four-part formula (Theorem 16.3):

$$\begin{aligned}\cos(90-\lambda)\cos\epsilon &= \cot 90^\circ \sin\epsilon - \cot(90+\alpha)\sin(90-\lambda)\\ \sin\lambda\cos\epsilon &= \tan\alpha\cos\lambda\\ \tan\alpha &= \tan\lambda\cos\epsilon\\ \tan\alpha_a &= \tan(\omega(T-T_g))\cos\epsilon^\circ\end{aligned}$$

Since the arctan function limits us to $\alpha, \lambda \in [-90^\circ, 90^\circ]$, we use a piecewise function obtain a function for $\lambda \in [0^\circ, 360^\circ)$:

$$\alpha_a(T) = \begin{cases} \arctan[\cos\epsilon^\circ \tan(\omega(T-T_g))] & \text{for } \alpha, \lambda \in [0^\circ, 90^\circ] \\ \arctan[\cos\epsilon^\circ \tan(\omega(T-T_g))] + 180^\circ & \text{for } \alpha, \lambda \in (90^\circ, 270^\circ) \\ \arctan[\cos\epsilon^\circ \tan(\omega(T-T_g))] + 360^\circ & \text{for } \alpha, \lambda \in [270^\circ, 360^\circ) \end{cases} \tag{16.12}$$

The equation of time is then given by Equation 16.11:

$$EoT = \omega(T-T_g) - \alpha_a(T)$$

If you plot this on a graphing software, you will see that it is a sinusoidal shape with a period of half a year (see Figure 16.22, blue) and amplitude $2.48^\circ \approx 10$ minutes, with zeroes at the equinoxes and solstices. At the equinoxes ($\omega(T-T_g) = 0^\circ$ and $\omega(T-T_g) = 180^\circ$), the equation of time is increasing; at the solstices ($\omega(T-T_g) = 90^\circ$ and $\omega(T-T_g) = 270^\circ$), the equation of time is decreasing. Conceptually, this can be understood with the fact that as the apparent sun travels along the ecliptic, its motion's right ascension (East / West) and declination (North / South) components will vary. At the solstices, the sun travels parallel to the ecliptic, so all of its motion is in the right ascension direction: the apparent sun's right ascension increases relative to the mean sun ($\frac{d\alpha_a}{dt} > \frac{d\alpha_m}{dt}$)[3], and the equation of time decreases. Similarly, at the equinoxes, the sun has a significant component of motion that is in the North / South direction: the apparent sun's right ascension decreases relative to the mean sun ($\frac{d\alpha_a}{dt} < \frac{d\alpha_m}{dt}$), and the equation of time increases.

Now that we have analyzed these two effects in turn, all that is left is to superimpose them, yielding a lopsided oscillating function of amplitude $\sim 15$ minutes.

An interesting consequence of the equation of time is the **solar analemma**. If you were to take a picture of the sun every day at the same clock time (ignoring Daylight Savings) and track its apparent motion, you would see that it makes a figure 8 shape throughout the year.

We can split the analemma into its North/South and East/West components. The sun moves North/South throughout the year because it changes in declination (due to Earth's obliquity, see Figure 16.10). If the Northern Hemisphere is tilted towards the sun, the sun will appear to have a higher declination, and if the Southern Hemisphere is tilted towards the sun, the sun

[3] Why aren't they equal? The reason is because the sun (at the solstices) is not traveling on the equator. For the same change in right ascension, stars farther from the celestial equator will travel along smaller arc lengths; this ratio is $\sin\delta$. As both the mean and apparent suns are traveling with the same angular speed, the change in right ascension is greater for the object farther from the equator for the same arc length travelled.

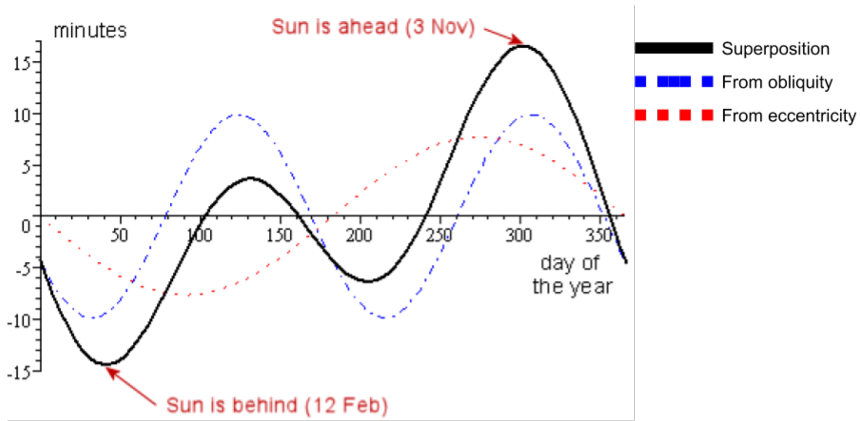



Figure 16.22: The equation of time. The contributions from Earth's axial tilt and eccentric orbit are given in blue and red, respectively. The resulting superposition is an uneven function oscillating between a faster and slower apparent solar time. (Image Credit: IntMath)

will have a lower (more negative) declination. The height of the figure 8 is simply the range of this variation, which is twice Earth's obliquity. On the other hand, the East/West movement is caused by the equation of time. There is a fantastic animation illustrating the relationship between the equation of time and the sun's movement on the analemma on the Wikipedia page for the Equation of Time, and I strongly suggest checking it out.

Lastly, I would like to briefly discuss the actual time systems we use, which is slightly more complicated than just local mean solar time. Our clock times are standardized to time zones. The Earth is divided into 24 different time zones, each with 15° of longitude, although some regions do not adhere to these time zones. These time zones are designated either by name (CST, UT, GMT, etc.) or number. For example, the Pacific Standard Time zone (PST) is GMT-7, which means it is 7 hours behind Greenwich Mean Time. Time in these time zones are standardized in the following way: Coordinated Universal Time (UTC), standardized by atomic pulses, is the standard for all time zones, and is set to be close to the local mean solar time on the Greenwich Meridian. Greenwich Mean Time (GMT) is the time zone for [-7.5, 7.5] longitude, so it is equal to UTC. Note that UTC is a time standard while GMT is a time zone, and both can be used interchangeably to designate the local mean solar time of 0° longitude. Other time zones are then integer hours offset from UTC or GMT. Because of this system, the time of GMT+1 ([7.5, 22.5]) is equal to the local mean solar time of 15 longitude, and the time of GMT+2 [22.5, 37.5] is equal to the local mean solar time of 30 longitude. Notice that time zones are generally standardized to the local mean solar time of the longitude in the middle of the time zone. Also note that many areas do not adhere to these zones: all of China is in one time zone (GMT+8, [112.5°, 127.5°]) even though China extends as far West as 80° E longitude. Keep in mind that you must correct for these differences when solving a problem that involves the official clock time!

Figure 16.23: The solar analemma. Here, a picture of the sun was taken every day at noon mean solar time, so it is symmetric about the meridian. (Image Credit: Universe Today)

Let's wrap up this chapter with one last problem...

**Example 16.8.** After crash landing on the Japanese island of Aoshima (33.7°N, 132.5°E) on the vernal equinox, Robert the astronomer is now awaiting to be rescued. In the meantime, he decides to continue his studies of the sun. Having now set his watch to the local time (GMT+9), he climbs a nearby mountain, intending to record the time of sunset. He watches the sunset at the summit, which is 970.5 m above sea level. At what time will he see the sun set, to the nearest minute? Note that sunset is when the *entirety* of the sun, not just its center, is below the horizon. The Earth has a radius of 6371 km, and the sun has an angular radius of 0.5°. On this vernal equinox, the equation of time $EoT = t_a - t_m =$ -7m 36s.

*Solution.* On the vernal equinox, the sun's declination is 0°. Thus, the sun travels on a great circle, rising at true east and setting at true west, and sets exactly 6 hours after transiting. (You may verify this with a coordinate conversion formula.) Unfortunately, there are a few details that prevent us from simply answering 6 PM:

1. Robert is at the top of a mountain, which means he can see slightly below the horizon (i.e. he can see slightly below a zenith distance of 90°). This angle is equal to

$$\arccos[R_\oplus/(R_\oplus + 0.9705\,\text{km})] = 1^\circ.$$

2. Even if Robert were not at the top of the mountain, it is the *center* of the sun that crosses the horizon 6 hours after transiting. We will have to find when the sun reaches -0.5°, the angular radius of the sun, below Robert's horizon.

3. The equation of time is -7m 36s, which means the real sun transits *after* the mean sun at 12:07:36 PM local mean solar time.

4. Local mean solar time is not equal to clock time. Robert's watch is set to GMT+9, which is the local mean solar time of $15^\circ/\text{hour} \times 9\,\text{hours} = 135^\circ$ E. However, he is at longitude 132.5°E.

Let's start with finding the hour angle of the sun at sunset; we can then find the time that the sun transits in GMT+9 and add the two to find the sunset time.

Because Robert is on top of a mountain, he can see 1° below the horizon. Furthermore, the sun has an angular radius of 0.5°. Therefore, the center of the sun must have an altitude of -1.5° in order for it to be completely out of Robert's view. We can use Equation 16.10 to solve for the hour angle of the sun at this altitude. Recall that the sun is at the vernal equinox on this day and thus has equatorial coordinates $(\alpha, \delta) = (0^h, 0^\circ)$.

$$\begin{aligned}\cos H &= \frac{\sin a - \sin\delta\sin\phi}{\cos\delta\cos\phi} \\ &= \frac{\sin -1.5^\circ - \sin 0^\circ \sin 33.7^\circ}{\cos 0^\circ \cos 33.7^\circ} = -0.0315\end{aligned}$$

The sun is setting, so the hour angle $H = \arccos = 0.0315 = 91.8^\circ = 6\text{h } 7\text{m } 13\text{s}$ is positive. Next, let's find the clock time at which the apparent sun transits. From the equation of time, we know that the apparent sun transits 7 minutes and 36 seconds after the mean sun transits at 12 PM local mean solar time. We can convert local mean solar time to clock time with the fact that clock time is simply the local mean solar time of longitude 135°E, 2.5° east of Aoshima. Clock time is thus $2.5^\circ = 10$ minutes ahead of Aoshima's local mean solar time, and 12:07:36 PM local mean solar time corresponds to 12:17:36 PM clock time.

The sun sets 6h 7m 13s after 12:17:36 PM, or at 6:25 PM.

□

## 16.5 Problems

**Problem 16.1.** If Vega is culminating in San Diego, California (latitude 32.7157° N, longitude 117.1611° W), what is its hour angle in Cambridge, Massachusetts (latitude 42.3601° N, longitude 71.0942° W)?

**Problem 16.2.** If Vega culminates at midnight CST (GMT-6) in Lubbock, Texas (latitude 33.5779° N, longitude 101.8552° W), what time (in PST) will it culminate in San Diego, California (latitude 32.7157° N, longitude 117.1611° W)? Note that PST is GMT-8.

**Problem 16.3.** Calculate the angular distance between Betelgeuse ($\alpha$ *Ori*, $\alpha = 05^h55^m10.3^s$, $\delta = +07°24'25.4$”) and Antares($\alpha$ *Sco*, $\alpha = 16^h29^m24.5^s$, $\delta = -26°25'55.2$”).

**Problem 16.4 (2018 USAAAO).** From which geographic latitude does the star Antares ($\alpha$ Scorpio, $\delta = 26°\ 190'$) never rise?

A. $26°\ 19'$ B. $63°\ 41'$ C. $56°\ 19'$ D. Never happens E. $53°\ 41'$

**Problem 16.5 (2018 USAAAO).** For the following problem, find the range in which the answer lies: looking from Greenwich on February 10th ($LST - t_{\text{clock}} = 9^h17^m48^s$), at what time is Pollux ($\alpha = 7^h53^m16^s$ at its upper culmination?

**Problem 16.6 (2019 USAAAO).** The celestial coordinates of the Orion Nebula are RA $0^h35^m$, Dec $05°23'$. Which of the following is closest to the time (local solar time) when the Orion Nebula would cross the meridian on the night of February 1st 2019? The date of the vernal equinox of 2019 is March 20th.

A. 8:40 PM B. 10:22 PM C. 12:00 AM D. 01:38 AM E. 03:20 AM

**Problem 16.7 (2018 NAO).** On March 21st at true noon, length of the shadow of a vertical rod was equal to its height. On which geographic latitude did this happen?

**Problem 16.8 (2019 NAO).** You are in the northern hemisphere and are observing rise of star A with declination $\delta = -8°$, and at the same time a star B with declination $\delta = +16°$ is setting. What will happen first: the next setting of the star A or rising of the star B?

**Problem 16.9.** Prove that the hour angle $H$ of a setting star is given by the expression

$$\cos H = -\tan\phi\tan\delta$$

where $\phi$ is the observer's latitude and $\delta$ is the star's declination.

**Problem 16.10.** Prove that the equation of the ecliptic in equatorial coordinates ($\alpha$, $\delta$) has the form:

$$\tan\delta = \sin\alpha\tan\epsilon$$

where $\epsilon = 23.45°$ is Earth's obliquity.

**Problem 16.11 (IOAA 2009).** Damavand Mountain is located in the northern part of Iran, on the south coast of the Caspian Sea. Consider an observer standing on top of Damavand Mountain (latitude $= 35°57'$ N; longitude $= 52°6'$ E; altitude $5.6\times10^3$ m from mean sea level) and looking at the sky over the Caspian Sea. What is the minimum declination for a star to be seen marginally circumpolar for this observer? The surface level of the Caspian Sea is approximately equal to the mean sea level.

**Problem 16.12 (IOAA 2007).** For an observer at latitude 42.5° N and longitude 71° W, estimate the time of sunrise on 21 December if the observer's civil time is -5 hours from GMT. Ignore refraction by the atmosphere, the size of the solar disc, and the equation of time.

**Problem 16.13 (IOAA 2010).** Find the equatorial coordinates (hour angle and declination) of a star at Madrid (geographic latitude $\phi = 40°$ at the instant when the star is at zenith angle $z = 30°$ and azimuth $A = 50°$ (azimuth is measured eastward from true south).

**Problem 16.14 (IOAA 2010).** What is the hour angle $H$ and the zenith angle $z$ of Vega ($\delta = 38°47'$) in Thessaloniki ($\lambda_1 = 1^h 32^m$, $\phi_1 = 40°37'$), at the moment it culminates at the local meridian of Lisbon ($\lambda_2 = -0^h 36^m$, $\phi_1 = 39°43'$)?

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